Compact Template for Polynomial Division

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***Abstract*-A compact template for the division of two giving polynomials to find quotient and reminder is derived. The process is very simple and direct, comparing to the familiar classical long polynomial division and synthetic polynomial division.**

Keywords-Polynomial division; Long polynomial division; Synthetic polynomial division; Convolution matrix; Polynomial multiplication.

I. INTRODUCTION

There are several approaches for finding the quotient and remainder from dividing two given univariate polynomials. Long polynomial division is very popular but tedious in computation, and widely used even by high school students. Synthetic polynomial division is fairly easy to use but only appropriate for the linear divisor [1]. Convolution polynomial division [2] is direct in operation, and used in MATLAB built-in routine.

This work presents a compact template for polynomial division. The process is very simple and straightforward and does not need to write down any intermediate steps, as in the familiar classical long polynomial division and synthetic polynomial division. It is extremely suitable for hand computation with a plain calculator.

The concept of compact template may be easily applied to the product of two given univariate polynomials.

Two typical numerical examples are provided to show the merit of the approach presented

II. POLYNOMIAL DIVISION

The division of two given polynomials is expressed as

$$\frac{b(x)}{a(x)} = q(x) + \frac{r(x)}{a(x)}$$

where the dividend b(x) of degree n and divisor a(x) of degree m are

$$b(x) = b_0 x^n + b_1 x^{n-1} + \cdots\cdots + b_{n-1} x + b_n$$
$$a(x) = a_0 x^m + a_1 x^{m-1} + \cdots\cdots + a_{m-1} x + a_m$$

Then the quotient q(x) of degree n-m and remainder r(x) of degree m-1 are obtained,

$$q(x) = q_0 x^{n-m} + q_1 x^{n-m-1} + \cdots\cdots + q_{n-m-1} x + q_{n-m}$$
$$r(x) = r_0 x^{m-1} + r_1 x^{m-2} + \cdots\cdots + r_{m-2} x + r_{m-1}$$

By carefully examine the familiar classical long polynomial division we may find the following relations among the coefficients of these polynomials:

$$b_k = q_0 a_k + q_1 a_{k-1} + \cdots\cdots + q_{k-1} a_1 + q_k a_0, \quad k = 0, 1, \cdots, n-m$$

$$b_k - r_{k-(n-m+1)} = q_0 a_k + q_1 a_{k-1} + \cdots\cdots + q_{n-m} a_{k-(n-m)}, \quad k = n-m+1, \cdots, n$$

The computation process can then be made by the following "compact polynomial division template"

	b_0	b_1	$\cdots$	b_{n-m}	b_{n-m+1}	$\cdots$	$\cdots$	$\cdots$	b_n
÷)	a_0	a_1	$\cdots$	a_{n-m}	a_{n-m+1}	$\cdots$	a_m		
1	q_0	q_1	$\cdots$	q_{n-m}	r_0	r_1	$\cdots$	$\cdots$	r_{m-1}

and the desired coefficients are therefore calculated from the given coefficients in this template,

$$q_k = ([\,1\,]\cdot[b_k] - [\,q_0, \cdots, q_{k-1}\,]\cdot[\,a_k, \cdots, a_1\,]) \;/\; a_0, \quad k = 0, 1, \cdots, n-m$$

$$r_k = [\,1\,]\cdot[b_{n-m+1+k}] - [\,q_0, \cdots, q_{n-m}\,]\cdot[\,a_{n-m+1+k}, \cdots, a_{1+k}\,], \quad k = 0, 1, \cdots, m-1$$

Here $[\;\;]\cdot[\;\;]$ denotes the inner product of two equal size arrays; and it is clearly understood that $b_\ell = 0, \quad l > n$, and $a_\ell = 0, \quad l > m$.

The quick check can readily be achieved by the following simple operations:

$$a_m\, q_{n-m} = b_n - r_{m-1}$$

$$(a_0 + a_1 + \cdots + a_m)(q_0 + q_1 + \cdots + q_{n-m}) = (b_0 + b_1 + \cdots + b_n) - (r_0 + r_1 + \cdots + r_{m-1})$$

If all the given coefficients are integers and $a_0 \neq 1$, the scalar multiplier 1 in the template may be replaced by a^ (n-m+1), the desired coefficients are then obtained in pure arithmetic operations such that the round errors may be eliminated.

The compact polynomial division template for $n = 8,\ m = 5$ is shown for illustration:

	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
÷)	a_0	a_1	a_2	a_3	a_4	a_5			
1	q_0	q_1	q_2	q_3	r_0	r_1	r_2	r_3	r_4

The desired coefficients are computed in order:

$$q_0 = ([\,1\,]\cdot[b_0]) \;/\; a_0$$
$$q_1 = ([\,1\,]\cdot[b_1] - [\,q_0\,]\cdot[a_1\,]) \;/\; a_0$$
$$q_2 = ([\,1\,]\cdot[b_2] - [\,q_0 \;\; q_1\,]\cdot[\,a_2 \;\; a_1\,]) \;/\; a_0$$
$$q_3 = ([\,1\,]\cdot[b_3] - [\,q_0 \;\; q_1 \;\; q_2\,]\cdot[\,a_3 \;\; a_2 \;\; a_1\,]) \;/\; a_0$$

$$
\begin{aligned}
r_0 &= [1]\cdot[b_4] - [\,q_0\ \ q_1\ \ q_2\ \ q_3\,]\cdot[\,a_4\ \ a_3\ \ a_2\ \ a_1\,] \\
r_1 &= [1]\cdot[b_5] - [\,q_0\ \ q_1\ \ q_2\ \ q_3\,]\cdot[\,a_5\ \ a_4\ \ a_3\ \ a_2\,] \\
r_2 &= [1]\cdot[b_6] - [\,q_0\ \ q_1\ \ q_2\ \ q_3\,]\cdot[\,0\ \ a_5\ \ a_4\ \ a_3\,] \\
r_3 &= [1]\cdot[b_7] - [\,q_0\ \ q_1\ \ q_2\ \ q_3\,]\cdot[\,0\ \ 0\ \ a_5\ \ a_4\,] \\
r_4 &= [1]\cdot[b_8] - [\,q_0\ \ q_1\ \ q_2\ \ q_3\,]\cdot[\,0\ \ 0\ \ 0\ \ a_5\,]
\end{aligned}
$$

III. NUMERICAL EXAMPLES

Example1. The division of the two given polynomials:

$$b(x) = 4x^8 + 5x^7 - x^6 + 7x^5 - 6x^4 + x^3 + 2x^2 - 3x + 7$$

$$a(x) = 3x^5 + x^4 - 7x^3 + 5x^2 - 4x + 2$$

will gives the quotient and remainder:

$$q(x) = \frac{4}{3}x^3 + \frac{11}{9}x^2 + \frac{64}{27}x + \frac{176}{81}$$

$$r(x) = \frac{619}{81}x^4 + \frac{533}{81}x^3 - \frac{148}{81}x^2 + \frac{77}{81}x + \frac{215}{81}$$

The desired results may be determined either one of the following approaches:

From the comparison of the approaches in this example, the convolution polynomial division is obviously much simple and effective. The desired quotient and reminder are readily determined without computing any intermediate values as in the familiar classical longhand polynomial division and synthetic polynomial division.

1.Long polynomial division:

2.Synthetic polynomial division

3.Compact polynomial division template:

4.Compact template in pure integer operation:

(1)Long polynomial division:

$$
\begin{array}{r|ccccccccc}
 & +\frac{4}{3} & +\frac{11}{9} & +\frac{64}{27} & +\frac{176}{81} & & & & & \\
\hline
3+1-7+5-4+2\ ) & +4 & +5 & -1 & +7 & -6 & +1 & +2 & -3 & +7 \\
 & +4 & +\frac{4}{3} & -\frac{28}{3} & +\frac{20}{3} & -\frac{16}{3} & +\frac{8}{3} & & & \\
\cline{3-10}
 & & +\frac{11}{3} & +\frac{25}{3} & +\frac{1}{3} & -\frac{2}{3} & -\frac{5}{3} & +2 & -3 & +7 \\
 & & +\frac{11}{3} & +\frac{11}{9} & -\frac{77}{9} & +\frac{55}{9} & -\frac{44}{9} & +\frac{22}{9} & & \\
\cline{4-10}
 & & & +\frac{64}{9} & +\frac{80}{9} & -\frac{61}{9} & +\frac{29}{9} & -\frac{4}{9} & -3 & +7 \\
 & & & +\frac{64}{9} & +\frac{64}{27} & -\frac{448}{27} & +\frac{320}{27} & -\frac{256}{27} & +\frac{128}{27} & \\
\cline{5-10}
 & & & & +\frac{176}{27} & +\frac{265}{27} & -\frac{233}{27} & +\frac{244}{27} & -\frac{209}{27} & +7 \\
 & & & & +\frac{176}{27} & +\frac{176}{81} & -\frac{1232}{81} & +\frac{880}{81} & -\frac{704}{81} & +\frac{352}{81} \\
\cline{6-10}
 & & & & & +\frac{619}{81} & +\frac{533}{81} & -\frac{148}{81} & +\frac{77}{81} & +\frac{215}{81}
\end{array}
$$

2)Synthetic polynomial division

	+4	+5	−1	+7	−6	+1	+2	−3	+7
$-\frac{1}{3}$		$-\frac{4}{3}$	$-\frac{11}{9}$	$-\frac{64}{27}$	$-\frac{176}{81}$				
$+\frac{7}{3}$			$+\frac{28}{3}$	$+\frac{77}{9}$	$+\frac{448}{27}$	$+\frac{1232}{81}$			
$-\frac{5}{3}$				$-\frac{20}{3}$	$-\frac{55}{9}$	$-\frac{320}{27}$	$-\frac{880}{81}$		
$+\frac{4}{3}$					$+\frac{16}{3}$	$+\frac{44}{9}$	$+\frac{256}{27}$	$+\frac{704}{81}$	
$-\frac{2}{3}$						$-\frac{8}{3}$	$-\frac{22}{9}$	$-\frac{128}{27}$	$-\frac{352}{81}$
	+4	$+\frac{11}{3}$	$+\frac{64}{9}$	$+\frac{176}{27}$	$+\frac{619}{81}$	$+\frac{533}{81}$	$-\frac{148}{81}$	$+\frac{77}{81}$	$+\frac{215}{81}$

(3)Compact polynomial division template:

	+4	+5	−1	+7	−6	+1	+2	−3	+7
÷)	+3	+1	−7	+5	−4	+2			
1	$+\frac{4}{3}$	$+\frac{11}{9}$	$+\frac{64}{27}$	$+\frac{176}{81}$	$+\frac{619}{81}$	$+\frac{533}{81}$	$-\frac{148}{81}$	$+\frac{77}{81}$	$+\frac{215}{81}$

(4)Compact template in pure integer operation:

	+4	+5	−1	+7	−6	+1	+2	−3	+7
÷)	+3	+1	−7	+5	−4	+2			
81	+108	+99	+192	+176	+619	+533	−148	+77	+215

Example2. A given polynomial is to be expressed as powers of linear factor (2x -3):

$$p(x) = 2x^7 - 13x^6 + 51x^5 - 119x^4 + 191x^3 - 181x^2 + 104x + 6$$

$$p(x) = +\frac{1}{64}(2x-3)^7 + \frac{1}{8}(2x-3)^6 + \frac{57}{64}(2x-3)^5 + \frac{61}{16}(2x-3)^4 + \frac{811}{64}(2x-3)^3 + \frac{125}{4}(2x-3)^2 + \frac{3259}{64}(2x-3) + \frac{1125}{16}$$

by working the polynomial divisions consecutively:

		2	−13	+51	−119	+191	−181	+104	+6
÷)		+2	−3						
	1	+1	−5	+18	$-\frac{65}{2}$	$+\frac{187}{4}$	$-\frac{163}{8}$	$+\frac{343}{16}$	$+\frac{1125}{16}$
÷)		+2	−3						
	1	$+\frac{1}{2}$	$-\frac{7}{4}$	$+\frac{51}{8}$	$-\frac{107}{16}$	$+\frac{427}{32}$	$+\frac{629}{64}$	$+\frac{3259}{64}$	
÷)		+2	−3						
	1	$+\frac{1}{4}$	$-\frac{1}{2}$	$+\frac{39}{16}$	$+\frac{5}{16}$	$+\frac{457}{64}$	$+\frac{125}{4}$		
÷)		+2	−3						
	1	$+\frac{1}{8}$	$-\frac{1}{16}$	$+\frac{9}{8}$	$+\frac{59}{32}$	$+\frac{811}{64}$			
÷)		+2	−3						
	1	$+\frac{1}{16}$	$+\frac{1}{16}$	$+\frac{21}{32}$	$+\frac{61}{16}$				
÷)		+2	−3						
	1	$+\frac{1}{32}$	$+\frac{5}{64}$	$+\frac{57}{64}$					
÷)		+2	−3						
	1	$+\frac{1}{64}$	$+\frac{1}{8}$						
÷)		+2	−3						
	0	$+\frac{1}{64}$							

IV. POLYNOMIAL MULTIPLICATION

The concept of compact template may be easily applied to the polynomial multiplication. Let a(x) of degree m and b(x) of degree n be the given polynomials defined as before. The resulted polynomial p(x) of degree m+n is the multiplication of these two given polynomials:

$$p(x) = a(x) \cdot b(x)$$

or $$(p_0 x^{m+n} + p_1 x^{m+n-1} + \cdots + p_{m+n-1} x + p_{m+n})$$
$$= (a_0 x^m + a_1 x^{m-1} + \cdots + a_{m-1} x + a_m) \cdot (b_0 x^n + b_1 x^{n-1} + \cdots + b_{n-1} x + b_n)$$

The desired coefficients are then determined from the given coefficients,

$$p_k = a_0 b_k + a_1 b_{k-1} + \cdots + a_{k-1} b_1 + a_k b_0 ,\qquad k = 0, 1, \cdots, m+n$$

The computation process can be easily made by the following “compact polynomial multiplication template”

$$\begin{array}{cccccccccc} & a_0 & a_1 & \cdots & a_m & & & & & \\ \times) & b_0 & b_1 & \cdots & \cdots & \cdots & b_n & & & \\ \hline & p_0 & p_1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & p_0 \end{array}$$

and the desired coefficients are thus calculated directly from the given coefficients in this template,

$$p_k = [a_0, \cdots, a_k] \cdot [b_k, \cdots, b_0], \qquad k = 0, 1, \cdots, m+n$$

Here [ ] · [ ] denotes the inner product of two equal size arrays; and it is clearly understood that $b_\ell = 0, \quad l > n,$ and $a_\ell = 0, \quad l > m$.

V. CONCLUSION

The useful template is derived for division of polynomials. By comparison from the examples, this approach is simple and effective. The desired quotient and reminder are readily determined without writing down any intermediate values as in the familiar classical longhand polynomial division and synthetic polynomial division.

One of the important applications is to find the roots with multiplicities of any given polynomial after the GCD of the polynomial and its derivative is computed [3-4].

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